Minimum and Approximate Minimum k-Cuts in Hypergraphs

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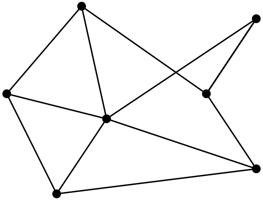

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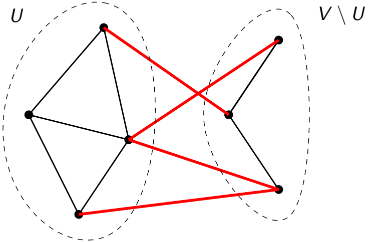

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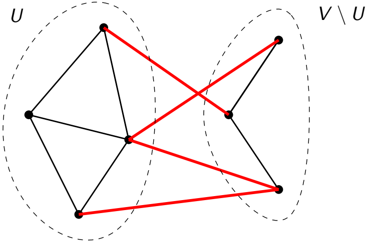

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Definition

An edge e crosses a cut U if it has one endpoint in U and the other in $V \setminus U$. The value of a cut is the number of edges that cross it.

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- Example application: Network reliability analysis
- Question: How can we efficiently find minimum and approximate minimum cuts?
- Question: How many minimum cuts can be in a graph with n vertices? How many α-approximate minimum cuts?

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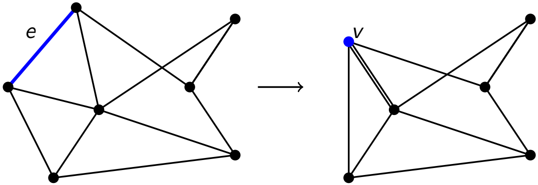

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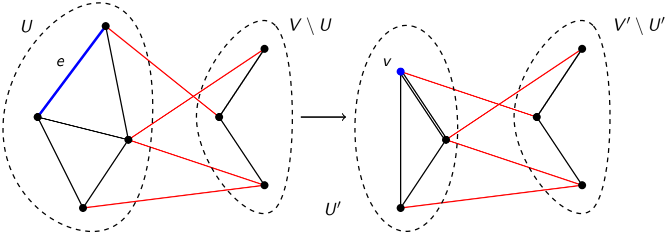

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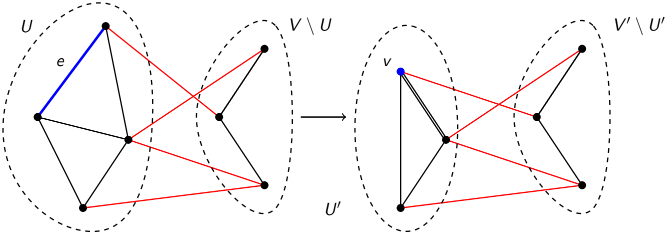

- The smaller a cut C, the more likely C survives the contraction of a randomly selected edge. In particular, a minimum cut is very likely to survive random contraction.

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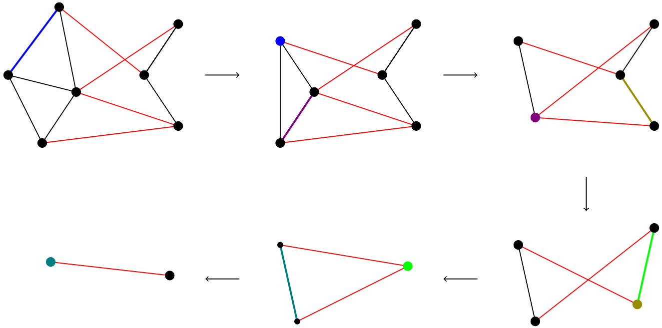

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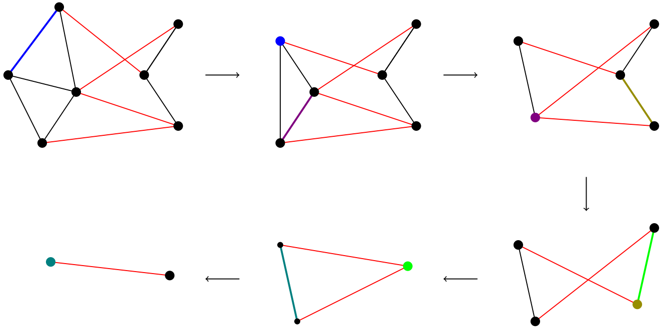

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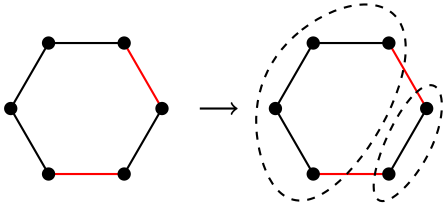
- The bound is tight: take a cycle graph on n vertices



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- First, let's consider k-cuts in graphs.

Definition

A *k-way cut* (or *k-cut* for short) is a partition of the vertices V into k non-empty subsets $U_1, U_2, \ldots, U_k$. An edge crosses a k-cut if its vertices are in different subsets U_i and U_j, and the size of a k-cut is the number of edges crossing it.

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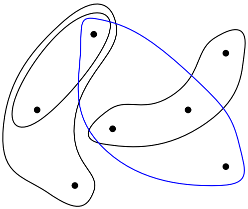
- Karger's algorithm readily generalizes to the k-cut problem - we stop contraction at k vertices remaining instead of two.

Hypergraphs

- In a graph, edges can connect only two vertices. In a hypergraph, we allow hyperedges to connect multiple vertices (i.e., each hyperedge $e \in E$ is a subset of V, so $E \subseteq 2^V$.)



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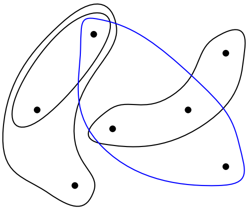

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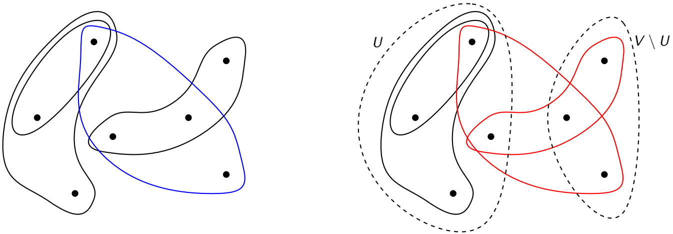

- We generalize the minimum cut and minimum k-cut problems to hypergraphs.

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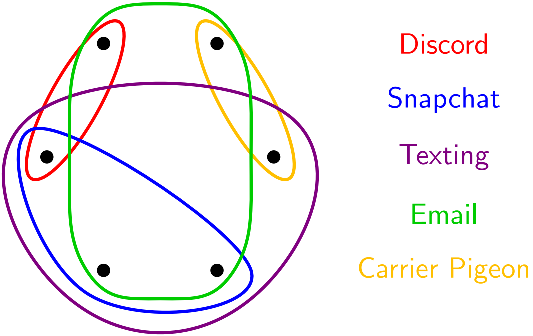
- Example: Communication platforms



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Theorem (Bao, Pan, Wang, and Zhao 2024+)

In a rank-r hypergraph H with n vertices, there are at most

$$O\left(k^{\alpha(k-1)r} n^{2\alpha(k-1)}\right)$$

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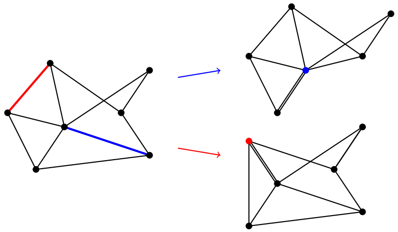

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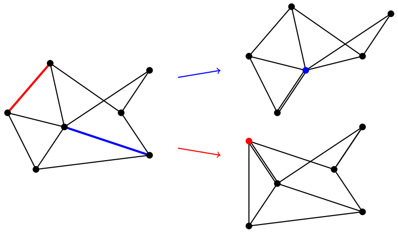

- Branches occur at predetermined times: every time $|V|$ decreases by a factor of $\sqrt{2}$.

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$$P_{\mathrm{Branch}}(e) = 1 - \frac{\binom{n-|e|}{k-1}}{\binom{n}{k-1}}$$

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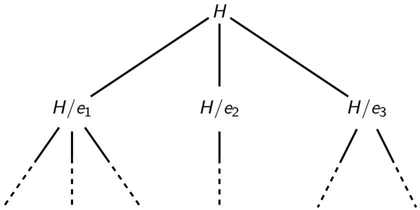

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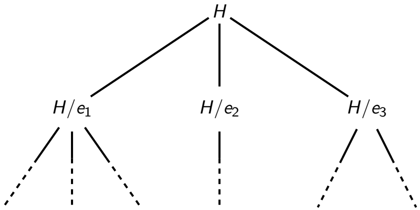

Theorem (Fox et al. 2018)

The branching contraction algorithm finds a minimum k-cut with high probability in $\tilde{O}(mn^{2k-2})$ expected time

Branching Contraction Time Complexity

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Theorem (Bao, Pan, Wang, and Zhao 2024+)

For an unweighted hypergraph without parallel edges, the branching contraction algorithms works in $\tilde{O}(mn^k + n^{2k})$ expected time.

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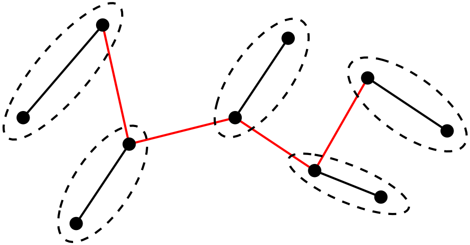

$$k = 5$$

$$|F| = 9$$

$$n = 10$$

$$|F| + k - n = 4$$

Fractional k-Cut

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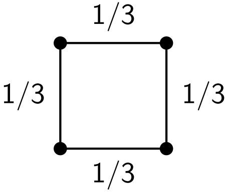

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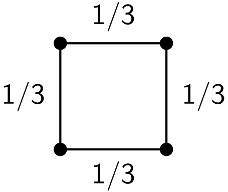

A fractional 2-cut smaller than the minimum 2-cut

In graphs, the minimum k-cut has value at most $2(1 - 1/n)$ times the minimum fractional k-cut. Furthermore, the minimum fractional k-cut is easier to compute.

Approximating Minimum k-cut Using Tree Packing

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- The fractional k-cut can be "rounded" to a $2(1+\varepsilon)$-approximate minimum k-cut.
- Take every edge with weight at least $1/2$. If this does not form a k-cut, then greedily complete with cuts from the minimum spanning tree (or forest).

Hypertrees

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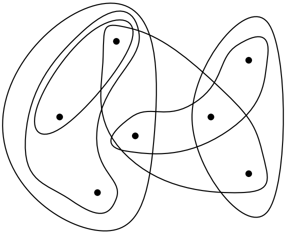

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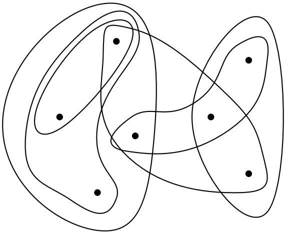

Theorem (Bao, Pan, Wang, and Zhao 2024+)

A minimum hypertree can be computed in $O(mn^2)$ time.

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Theorem (Bao, Pan, Wang, and Zhao 2024+)

There exists a $r(1+\varepsilon)$-approximation to minimum k-cut in

$$O(mn\log^2 n/\varepsilon^2 + mn^2)$$

time for hypergraphs, where r is the rank of the hypergraph.

Acknowledgements

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- We thank PRIMES-USA and the organizers for this opportunity.

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